Access to Science, Engineering and Agriculture: Mathematics 1 MATH00030 Chapter 4 Solutions

1. (a) This is not a function.
   For example, $f(-1)$ is not defined, since -1 does not lie in the codomain.

   (b) This is a function.
   Its domain is $\mathbb{R}^+$ and its codomain is $\mathbb{R}$.

   (c) This is a function.
   Its domain is $\mathbb{R}^+$ and its codomain is $\mathbb{R}$.

   (d) This is not a function.
   For example, $f(-1)$ is not defined, since $-(-1)^2 = -1$ does not lie in the codomain.

   (e) This is a function.
   Its domain is $\mathbb{R}^-$ and its codomain is $\mathbb{R}^+$.

   (f) This is not a function.
   For example, $f(1)$ is not defined, since $\sqrt{1} = 1$ does not lie in the codomain.

   (g) This is not a function.
   For example, $f(0)$ is not defined, since $0-1 = -1$ does not lie in the codomain.

2. The graphs of the functions in Question 1 are as follows.

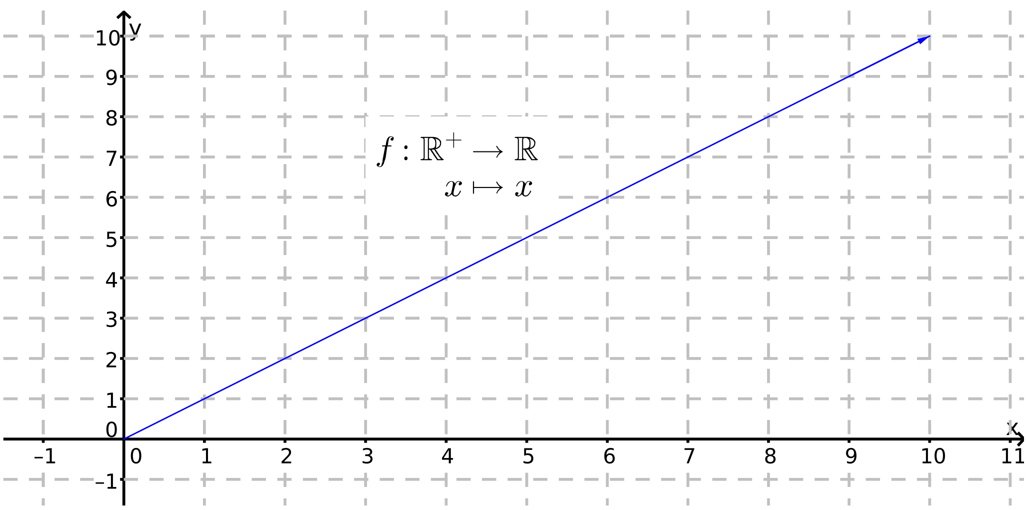

Figure 1: The Graph of the function defined in Question 1(b).

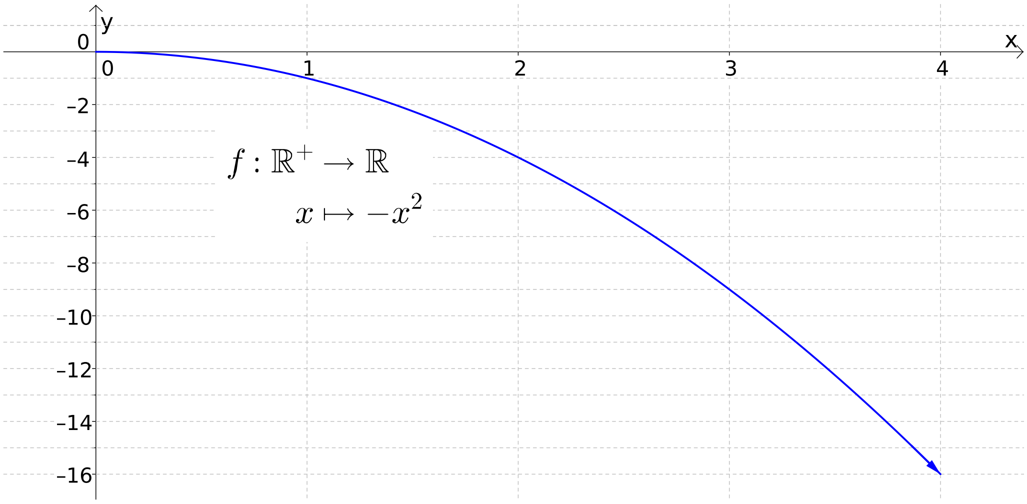

Figure 2: The Graph of the function defined in Question 1(c).

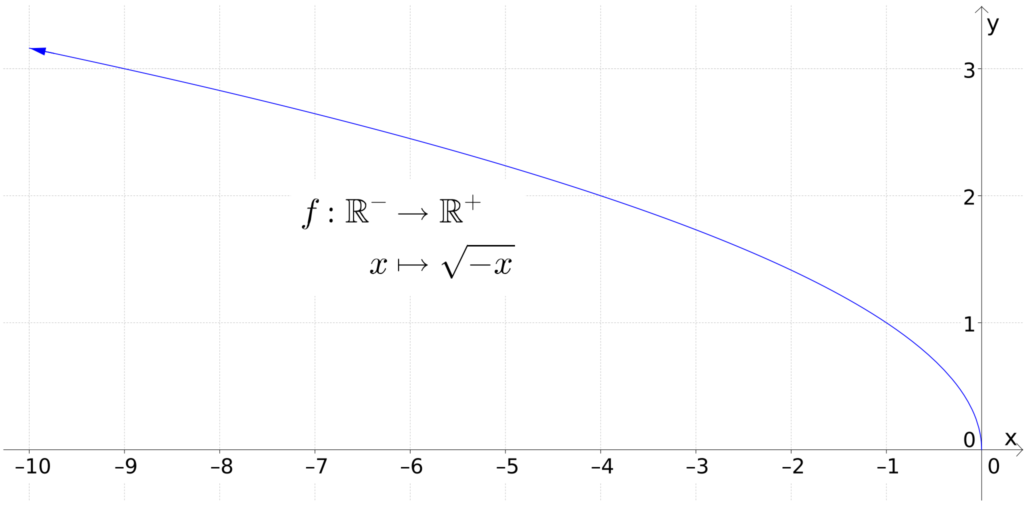

Figure 3: The Graph of the function defined in Question 1(e).

3. (a) Figure 4 shows the graph of the function

$$\begin{aligned} f\colon \{-36, -25, -16, -9, -4, 0\} &\to \mathbb{R}^+ \\ x &\mapsto \sqrt{-x} \end{aligned}$$

(b) Figure 5 shows the graph of the function

$$\begin{aligned} f\colon \{-4, -2, 0, 1, 4\} &\to \{0, 2, 4\} \\ -4 &\mapsto 0 \\ -2 &\mapsto 4 \\ 0 &\mapsto 2 \\ 1 &\mapsto 2 \\ 4 &\mapsto 4 \end{aligned}$$

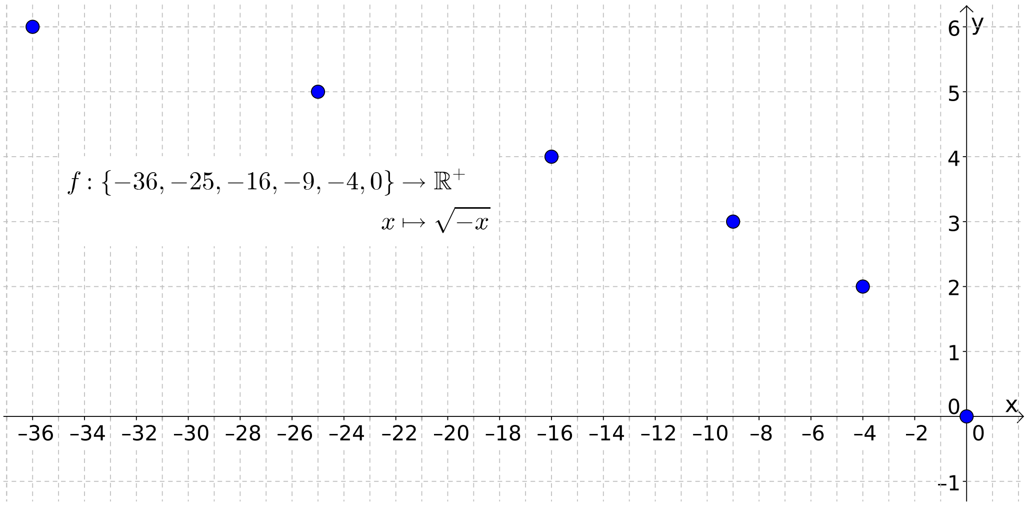

Figure 4: The Graph of the function defined in Question 3(a).

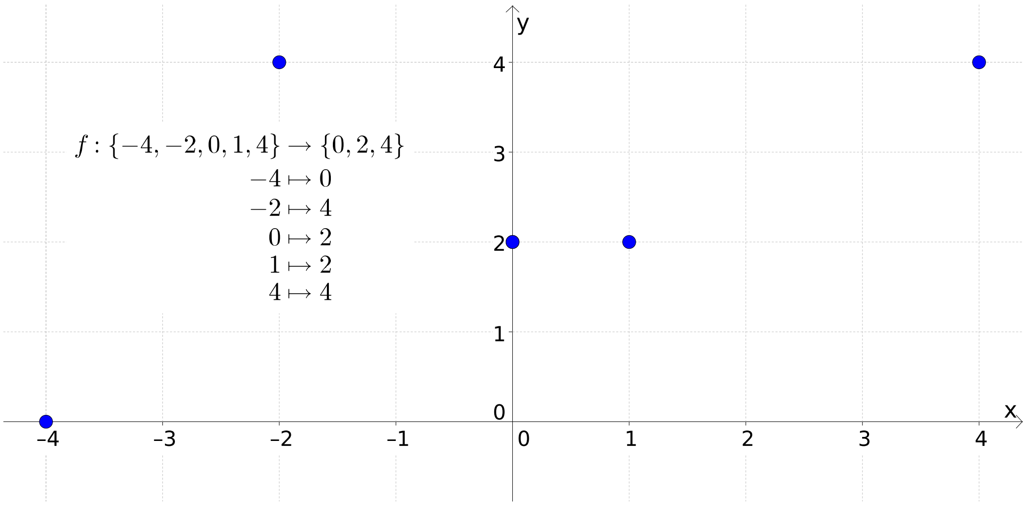

Figure 5: The Graph of the function defined in Question 3(b).

(c) Figure 6 shows the graph of the function

$$f\colon \{x \in \mathbb{R}\colon -1 \leqslant x \leqslant 2\} \to \{x \in \mathbb{R}\colon -1 \leqslant x \leqslant 7\}$$
$$x \mapsto 2x + 1$$

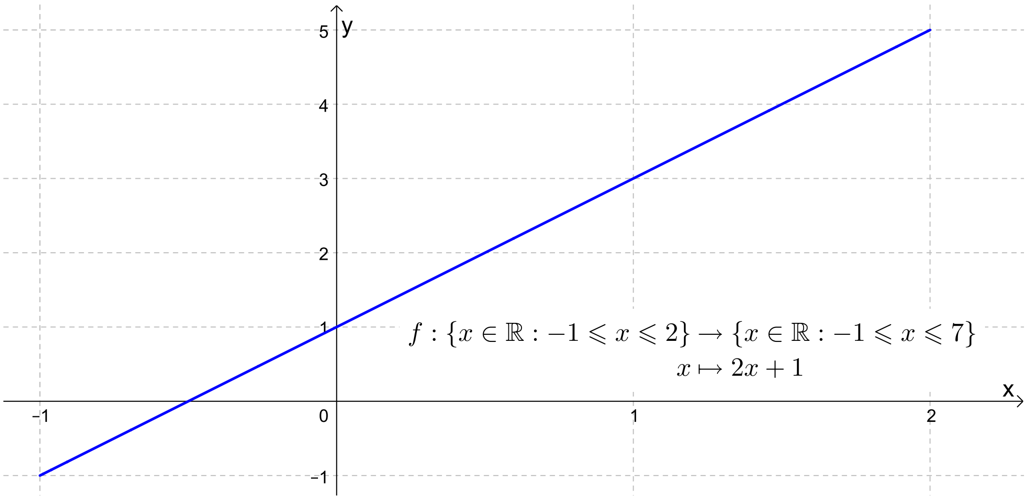

Figure 6: The Graph of the function defined in Question 3(c).

4. (a) This function is not injective since $f(A) = 2 = f(B)$.
   It is surjective.
   It is not bijective since it is not injective.

   (b) This function is not injective since $f(A) = 1 = f(C)$.
   It is not surjective since there is no x with $f(x) = 2$.
   It is not bijective since it is neither injective nor surjective.

   (c) This function is not injective since $f(B) = 2 = f(C)$.
   It is not surjective since there is no x with $f(x) = 3$.
   It is not bijective since it is neither injective nor surjective.

   (d) This function is injective.
   It is not surjective since there is no x with $f(x) = 1$.
   It is not bijective since it is not surjective.

   (e) This function is not injective since $f(C) = 1 = f(D)$.
   It is not surjective since there is no x with $f(x) = 4$.
   It is not bijective since it is neither injective nor surjective.

   (f) This function is injective, surjective and hence bijective.

   (g) This function is injective, surjective and hence bijective.

   (h) This function is injective.
   It is not surjective since there is no x with $f(x) = 0$.
   It is not bijective since it is not surjective.

   (i) This function is injective.
   It is not surjective since there is no x with $f(x) = -5$.
   It is not bijective since it is not surjective.

   (j) This function is injective.
   It is not surjective since there is no x with $f(x) = 1$.
   It is not bijective since it is not surjective.

   (k) This function is not injective since $f(1) = -2 = f(-1)$.
   It is surjective.
   It is not bijective since it is not injective.

(l) This function is injective, surjective and hence bijective.

5. The function in part (f) has inverse function

$$\begin{aligned} f^{-1}\colon \{1,2,3,4\} &\to \{A,B,C,D\} \\ 2 &\mapsto A \\ 3 &\mapsto B \\ 1 &\mapsto C \\ 4 &\mapsto D \end{aligned}$$

For the function in Part (g) we have $y = 3x - 4$, so

$$3x - 4 = y \Rightarrow 3x = y + 4 \Rightarrow x = \frac{1}{3}y + \frac{4}{3}.$$

Hence its inverse function is

$$\begin{aligned} f\colon \mathbb{R} &\to \mathbb{R} \\ y &\mapsto \frac{1}{3}y + \frac{4}{3} \end{aligned}$$

which can also be written as

$$\begin{aligned} f\colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \frac{1}{3}x + \frac{4}{3} \end{aligned}$$

For the function in Part (l) we have $y = -2x^2$, so

$$2x^2 = -y \Rightarrow x^2 = -\frac{y}{2} \Rightarrow x = -\sqrt{-\frac{y}{2}},$$

where we have taken the negative square root since the domain of f (the codomain of f^{-1}) is $\mathbb{R}^-$.
Hence its inverse function is

$$\begin{aligned} f\colon \mathbb{R}^- &\to \mathbb{R}^- \\ y &\mapsto -\sqrt{-\frac{y}{2}} \end{aligned}$$

which can also be written as

$$\begin{aligned} f\colon \mathbb{R}^- &\to \mathbb{R}^- \\ x &\mapsto -\sqrt{-\frac{x}{2}} \end{aligned}$$

6. We can eliminate $y = -4^x$ and $y = -\left(\frac{2}{3}\right)^x$ as possibilities since these functions are always negative and none of the graphs lie completely below the x-axis.
Since we have $\log_a(1) = 0$ for all $a > 0$ ($a \neq 1$), it follows that h and k must be (v) and (vi). Looking at Figure 15 in Chapter 4, we see that k is (v) (since $3 > 1$) and h is (vi) (since $0 < \frac{1}{4} < 1$).
Finally, looking at Figure 16 in Chapter 4, we see that f is (i) (since $3 > 1$) and g is (iv) (since $0 < \frac{3}{4} < 1$).
Summarizing: f is (i), g is (iv), h is (vi) and k is (v).

7. (a)

$$\begin{aligned} e^x = 5 \quad &\Longleftrightarrow \quad \ln(e^x) = \ln(5) \quad \text{taking the natural log of each side} \\ &\Longleftrightarrow \quad x = \ln(5) \quad \text{by Rule 8 of the Rules of Logs with } a = e. \end{aligned}$$

(b)

$$\begin{aligned} 4^x = 7 \quad &\Longleftrightarrow \quad \ln(4^x) = \ln(7) \quad \text{taking the natural log of each side} \\ &\Longleftrightarrow \quad x\ln(4) = \ln(7) \quad \text{by Rule 2 of the Rules of Logs} \\ &\Longleftrightarrow \quad x = \frac{\ln(7)}{\ln(4)} \quad \text{dividing each side by } \ln(4). \end{aligned}$$

(c)

$$-7^{3x} = 5 \quad \Longleftrightarrow \quad 7^{3x} = -5 \quad \text{multiplying both sides by } -1.$$

However the equation $7^{3x} = -5$ has no solutions since the exponent of a positive number (7 in this case) is always positive.
Thus the original equation $-7^{3x} = 5$ has no solutions either.

(d)

$$\begin{aligned} 10^{7x} = 3 \quad &\Longleftrightarrow \quad \log_{10}(10^{7x}) = \log_{10}(3) \quad \text{taking } \log_{10} \text{ of each side} \\ &\Longleftrightarrow \quad 7x = \log_{10}(3) \quad \text{by Rule 8 of the Rules of Logs with } a = 10 \\ &\Longleftrightarrow \quad x = \frac{\log_{10}(3)}{7} \quad \text{dividing each side by 7.} \end{aligned}$$

(e)

$$\begin{aligned} 9^{2x} = 8 \quad &\Longleftrightarrow \quad \ln\left(9^{2x}\right) = \ln(8) \quad \text{taking the natural log of each side} \\ &\Longleftrightarrow \quad 2x\ln(9) = \ln(8) \quad \text{by Rule 2 of the Rules of Logs} \\ &\Longleftrightarrow \quad x = \frac{\ln(8)}{2\ln(9)} \quad \text{dividing each side by } 2\ln(9). \end{aligned}$$

(f)

$$\begin{aligned} e^{-5x} = 4 \quad &\Longleftrightarrow \quad \ln(e^{-5x}) = \ln(4) \quad \text{taking the natural log of each side} \\ &\Longleftrightarrow \quad -5x = \ln(4) \quad \text{by Rule 8 of the Rules of Logs with } a = e \\ &\Longleftrightarrow \quad x = \frac{\ln(4)}{-5} \quad \text{dividing each side by } -5 \\ &\Longleftrightarrow \quad x = -\frac{\ln(4)}{5}. \end{aligned}$$

(g)

$$
\begin{aligned}
3^{-6x} = 2 \quad &\Longleftrightarrow \quad \ln(3^{-6x}) = \ln(2) \quad \text{taking the natural log of each side} \\
&\Longleftrightarrow \quad -6x\ln(3) = \ln(2) \quad \text{by Rule 2 of the Rules of Logs} \\
&\Longleftrightarrow \quad x = \frac{\ln(2)}{-6\ln(3)} \quad \text{dividing each side by } -6\ln(3) \\
&\Longleftrightarrow \quad x = -\frac{\ln(2)}{6\ln(3)}.
\end{aligned}
$$

(h)

$$
-9^{-5x} = 7 \quad \Longleftrightarrow \quad 9^{-5x} = -7 \quad \text{multiplying both sides by } -1.
$$

However the equation $9^{-5x} = -7$ has no solutions since the exponent of a positive number (9 in this case) is always positive.
Thus the original equation $-9^{-5x} = 7$ has no solutions either.

(i)

$$
\begin{aligned}
5\left(10^{-3x}\right) = 6 \quad &\Longleftrightarrow \quad 10^{-3x} = \frac{6}{5} \quad \text{dividing each side by 5} \\
&\Longleftrightarrow \quad \log_{10}(10^{-3x}) = \log_{10}\left(\frac{6}{5}\right) \quad \text{taking } \log_{10} \text{ of each side} \\
&\Longleftrightarrow \quad -3x = \log_{10}\left(\frac{6}{5}\right) \quad \text{by Rule 8 of the Rules of Logs with } a = 10 \\
&\Longleftrightarrow \quad x = \frac{\log_{10}(6/5)}{-3} \quad \text{dividing each side by } -3 \\
&\Longleftrightarrow \quad x = -\frac{\log_{10}(6/5)}{3}.
\end{aligned}
$$

(j)

$$
\begin{aligned}
-7\left(8^{-7x}\right) = -4 \quad &\Longleftrightarrow \quad 8^{-7x} = \frac{-4}{-7} \quad \text{dividing each side by } -7 \\
&\Longleftrightarrow \quad 8^{-7x} = \frac{4}{7} \\
&\Longleftrightarrow \quad \ln(8^{-7x}) = \ln\left(\frac{4}{7}\right) \quad \text{taking the natural log of each side} \\
&\Longleftrightarrow \quad -7x\ln(8) = \ln\left(\frac{4}{7}\right) \quad \text{by Rule 2 of the Rules of Logs} \\
&\Longleftrightarrow \quad x = \frac{\ln(4/7)}{-7\ln(8)} \quad \text{dividing each side by } -7\ln(8) \\
&\Longleftrightarrow \quad x = -\frac{\ln(4/7)}{7\ln(8)}.
\end{aligned}
$$

(k)

$$-6\left(5^{-2x}\right) = 5 \quad \iff \quad 6\left(5^{-2x}\right) = -5 \quad \text{multiplying both sides by } -1.$$

However the equation $6\left(5^{-2x}\right) = -5$ has no solutions since the exponent of a positive number (5 in this case) is always positive.
Thus the original equation $-6\left(5^{-2x}\right) = 5$ has no solutions either.

8. (a) (i) $30^\circ = 30 \times \dfrac{\pi}{180} = \dfrac{\pi}{6}$ Radians.

(ii) $135^\circ = 135 \times \dfrac{\pi}{180} = \dfrac{3\pi}{4}$ Radians.

(iii) $150^\circ = 150 \times \dfrac{\pi}{180} = \dfrac{5\pi}{6}$ Radians.

(iv) $330^\circ = 330 \times \dfrac{\pi}{180} = \dfrac{11\pi}{6}$ Radians.

(b) (i) $\dfrac{\pi}{4}$ Radians $= \left(\dfrac{180}{\pi} \times \dfrac{\pi}{4}\right)^\circ = 45^\circ$.

(ii) $\dfrac{\pi}{2}$ Radians $= \left(\dfrac{180}{\pi} \times \dfrac{\pi}{2}\right)^\circ = 90^\circ$.

(iii) $\dfrac{2\pi}{3}$ Radians $= \left(\dfrac{180}{\pi} \times \dfrac{2\pi}{3}\right)^\circ = 120^\circ$.

(iv) $\dfrac{7\pi}{4}$ Radians $= \left(\dfrac{180}{\pi} \times \dfrac{7\pi}{4}\right)^\circ = 315^\circ$.

9. (a) In this case we want to find $\sin(\theta)$ when $\theta = \dfrac{5\pi}{6}$.

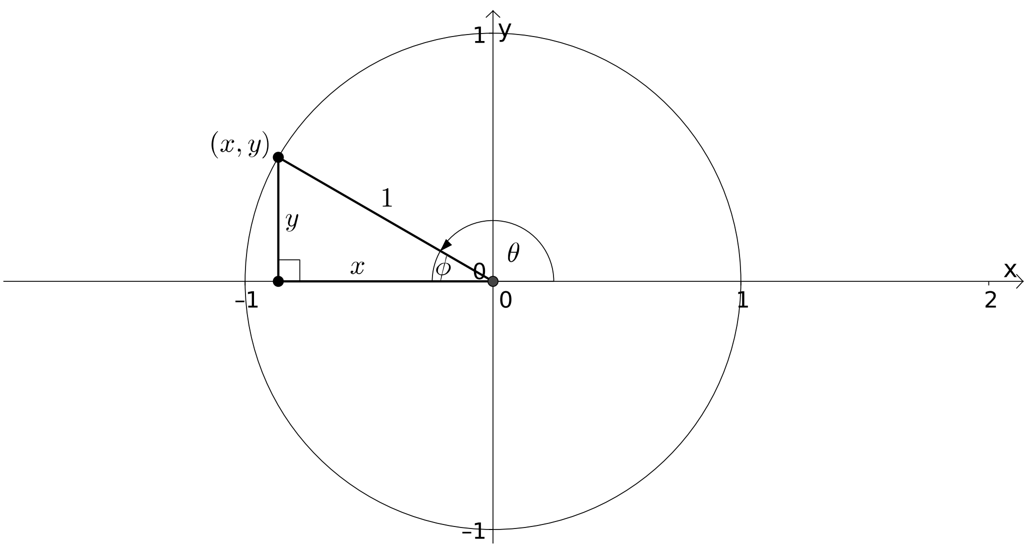

Figure 7: Calculation of $\sin\left(\dfrac{5\pi}{6}\right)$.

Looking at Figure 7, we see that we need to find y, since this is by definition $\sin\left(\dfrac{5\pi}{6}\right)$. Now, also from Figure 7, $\phi = \pi - \dfrac{5\pi}{6} = \dfrac{\pi}{6}$ (where we are just treating ϕ as an angle rather than a directed angle). Hence using Table 1 of Chapter 4, $\sin(\phi) = \frac{1}{2}$. But also by definition $\sin(\phi) = |y|$ (since the hypotenuse has length 1). Now, since y is positive, $y = |y|$ and so $\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$.

(b) In this case we want to find $\cos(\theta)$ when $\theta = \frac{7\pi}{6}$.

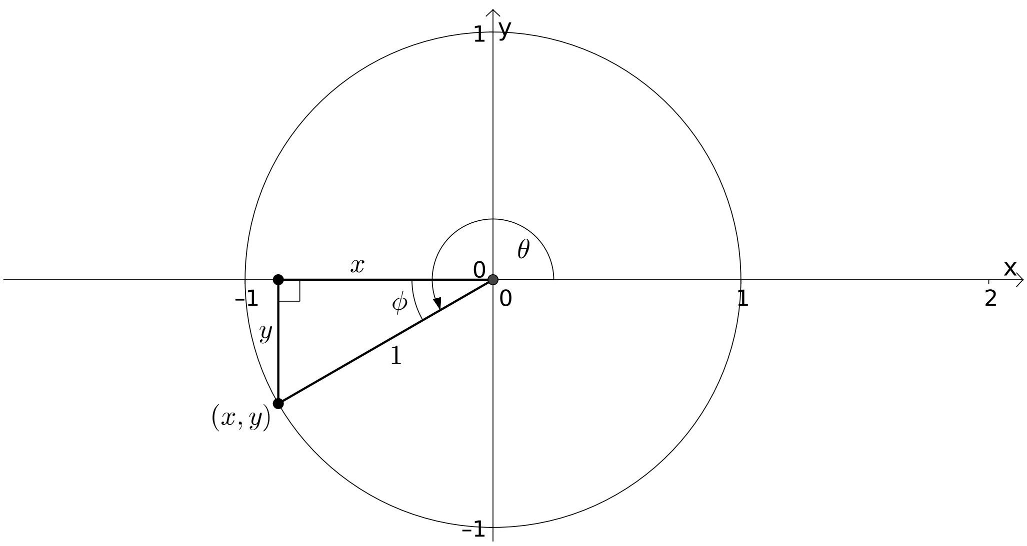

Figure 8: Calculation of $\cos\left(\frac{7\pi}{6}\right)$.

Looking at Figure 8, we see that we need to find x, since this is by definition $\cos\left(\frac{7\pi}{6}\right)$. Now, also from Figure 8, $\phi = \frac{7\pi}{6} - \pi = \frac{\pi}{6}$ (where we are just treating ϕ as an angle rather than a directed angle). Hence using Table 1 of Chapter 4, $\cos(\phi) = \frac{\sqrt{3}}{2}$. But also by definition $\cos(\phi) = |x|$ (since the hypotenuse has length 1). Now, since x is negative, $x = -|x|$ and so $\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$.

(c) In this case we want to find $\tan(\theta)$ when $\theta = \frac{5\pi}{4}$.

Looking at Figure 9, we see that we need to find $\frac{y}{x}$, since this is by definition $\tan\left(\frac{5\pi}{4}\right)$. Now, also from Figure 9, $\phi = \frac{5\pi}{4} - \pi = \frac{\pi}{4}$ (where we are just treating ϕ as an angle rather than a directed angle). Hence using Table 1 of Chapter 4, $\tan(\phi) = 1$. But also by definition $\tan(\phi) = \frac{|y|}{|x|}$. Looking at the diagram, we see that x and y are negative, so $\frac{y}{x} = \frac{|y|}{|x|}$. Hence $\tan\left(\frac{5\pi}{4}\right) = 1$.

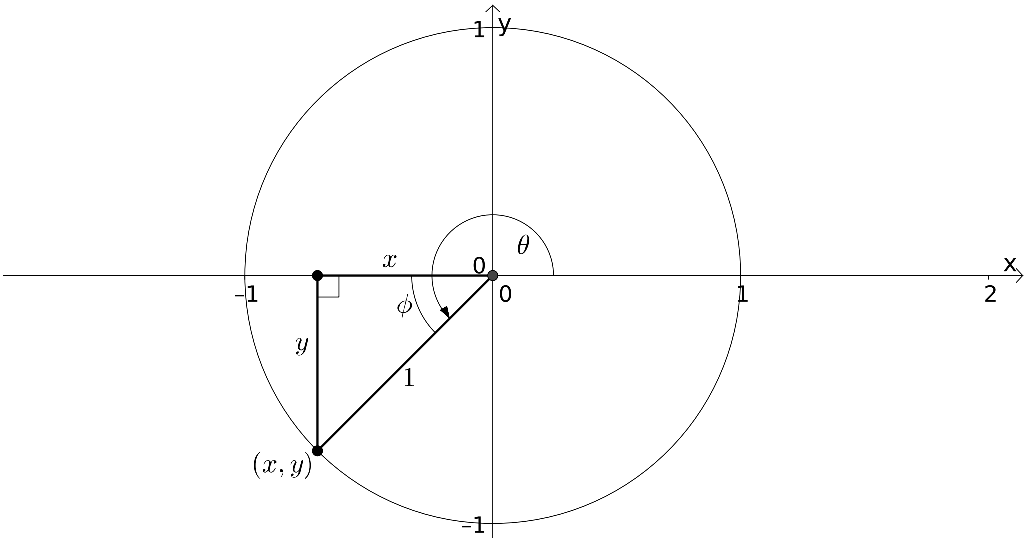

Figure 9: Calculation of $\tan\left(\frac{5\pi}{4}\right)$.

(d) In this case we want to find $\sin(\theta)$ when $\theta = -\frac{\pi}{4}$.

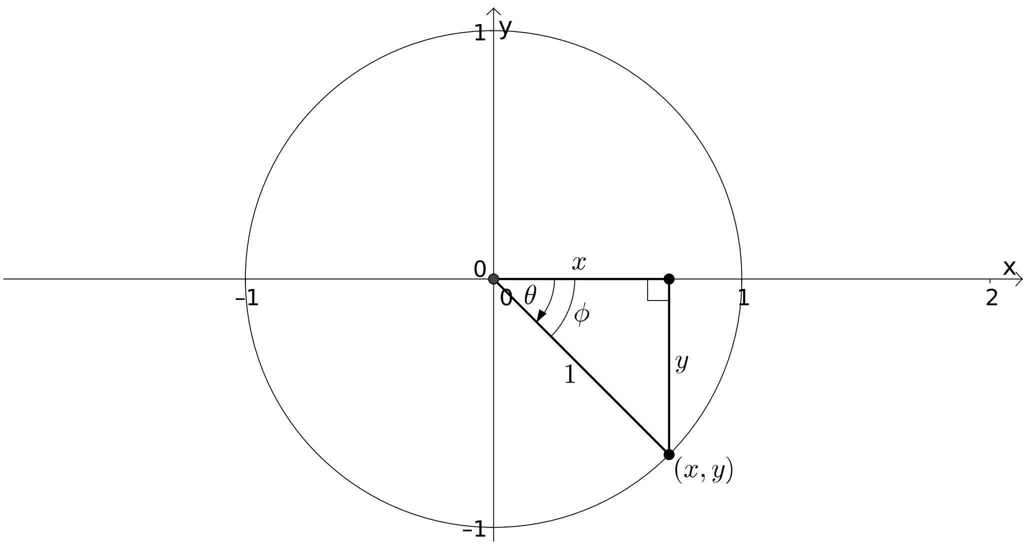

Figure 10: Calculation of $\sin\left(-\frac{\pi}{4}\right)$.

Looking at Figure 10, we see that we need to find y, since this is by definition $\sin\left(-\frac{\pi}{4}\right)$. Now, also from Figure 10, $\phi = |\theta| = \frac{\pi}{4}$ (where we are just treating ϕ as an angle rather than a directed angle). Hence using Table 1 of Chapter 4, $\sin(\phi) = \frac{1}{\sqrt{2}}$. But also by definition $\sin(\phi) = |y|$ (since the hypotenuse has length 1). Now, since y is negative, $y = -|y|$ and so $\sin\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$.

(e) To find $\operatorname{cosec}\left(\frac{4\pi}{3}\right)$, we will use $\operatorname{cosec}\left(\frac{4\pi}{3}\right) = \frac{1}{\sin\left(\frac{4\pi}{3}\right)}$. So we need to find $\sin(\theta)$ when $\theta = \dfrac{4\pi}{3}$.

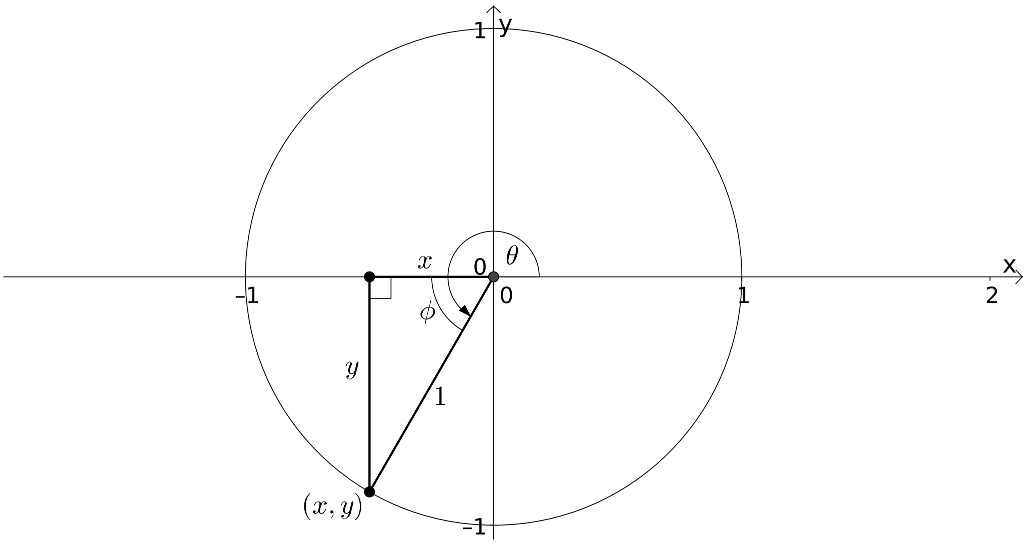

Figure 11: Calculation of $\operatorname{cosec}\left(\dfrac{4\pi}{3}\right) = \dfrac{1}{\sin\left(\dfrac{4\pi}{3}\right)}$.

Looking at Figure 11, we see that we need to find y, since this is by definition $\sin\left(\dfrac{4\pi}{3}\right)$. Now, also from Figure 11, $\phi = |\theta| = \dfrac{\pi}{3}$ (where we are just treating ϕ as an angle rather than a directed angle). Hence using Table 1 of Chapter 4, $\sin(\phi) = \dfrac{\sqrt{3}}{2}$. But also by definition $\sin(\phi) = |y|$ (since the hypotenuse has length 1). Now, since y is negative, $y = -|y|$ and so $\sin\left(\dfrac{4\pi}{3}\right) = -\dfrac{\sqrt{3}}{2}$. Hence $\operatorname{cosec}\left(\dfrac{4\pi}{3}\right) = \dfrac{1}{\sin\left(\dfrac{4\pi}{3}\right)} = \dfrac{1}{-\dfrac{\sqrt{3}}{2}} = -\dfrac{2}{\sqrt{3}}$.

(f) Here we will use the fact that the cotangent function repeats every 2π (in fact it repeats every π since it is the reciprocal of the tangent function but we don't need to use that here. Thus $\cot\left(\dfrac{9\pi}{4}\right) = \cot\left(\dfrac{9\pi}{4} - 2\pi\right) = \cot\left(\dfrac{\pi}{4}\right)$. Now $\cot\left(\dfrac{\pi}{4}\right) = \dfrac{1}{\tan\left(\dfrac{\pi}{4}\right)}$ and using Table 1 of Chapter 4, $\tan\left(\dfrac{\pi}{4}\right) = 1$. Hence $\cot\left(\dfrac{\pi}{4}\right) = \dfrac{1}{1} = 1$ and so we also have $\cot\left(\dfrac{9\pi}{4}\right) = 1$.

(g) Here we will first use the fact that the cosine function repeats every 2π. Hence $\cos\left(\dfrac{38\pi}{3}\right) = \cos\left(\dfrac{38\pi}{3} - 6 \times 2\pi\right) = \cos\left(\dfrac{2\pi}{3}\right)$, so we have to find $\cos\left(\dfrac{2\pi}{3}\right)$. So we want to find $\cos(\theta)$ when $\theta = \dfrac{2\pi}{3}$. Looking at Figure 12, we see that we need to find x, since this is by definition $\cos\left(\dfrac{2\pi}{3}\right)$. Now, also from Figure 12, $\phi = \pi - \frac{2\pi}{3} = \frac{\pi}{3}$ (where we are just treating ϕ as an angle rather than a directed angle). Hence using Table 1 of Chapter 4, $\cos(\phi) = \frac{1}{2}$. But also by definition $\cos(\phi) = |x|$ (since the hypotenuse has length 1). Now, since x is negative, $x = -|x|$ and so $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$. Hence we also have $\cos\left(\frac{38\pi}{3}\right) = -\frac{1}{2}$.

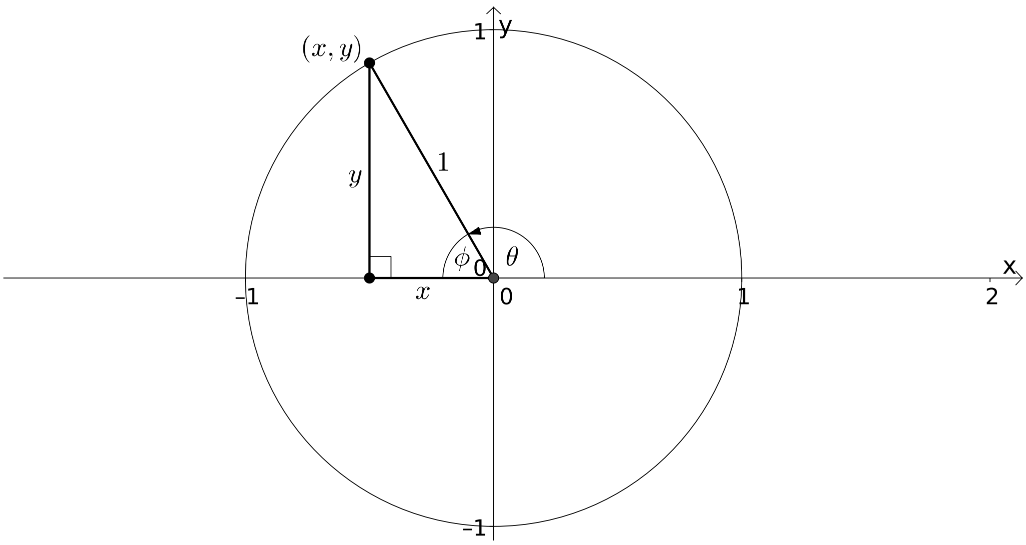

Figure 12: Calculation of $\cos\left(\frac{38\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right)$.